4103. Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.

Let x, y and z be positive numbers such that $x+y+z=1$. Show that

$$\sum_{cyc}(1-x)\sqrt{3yz(1-y)(1-z)} \geq 4\sqrt{xyz}$$

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In homogenous form original inequality is

$$\sum_{cyc}(y+z)\sqrt{3yz(x+y)(z+x)} \geq 4(x+y+z)\sqrt{(x+y+z)xyz}\ .$$

Let $a := y+z,\ b := z+x,\ c := x+y$. Then numbers a,b,c determine a triangle with sidelengths $a,b,c,$ semiperimeter $s = x+y+z,$ area $F = \sqrt{(x+y+z)xyz}$ and, therefore, original inequality equivalent to the following geometric inequality

(1) $\qquad \sum_{cyc} a\sqrt{3(s-b)(s-c)bc} \geq 4sF.$

Since $a\sqrt{(s-b)(s-c)bc} = abc\sqrt{\dfrac{(s-b)(s-c)}{bc}} = abc\sin\dfrac{A}{2} = 4RF$ then **(1)**$\Leftrightarrow$

(2) $\qquad \sum_{cyc}\sin\dfrac{A}{2} \geq \dfrac{s}{R\sqrt{3}}$

or, **(1)**$\Leftrightarrow \sum_{cyc}\dfrac{a}{\cos\frac{A}{2}} \geq \dfrac{4s}{\sqrt{3}}$, because $4R\sin\dfrac{A}{2} = \dfrac{2R\sin A}{\cos\frac{A}{2}} = \dfrac{a}{\cos\frac{A}{2}}$.

$$\sum_{cyc}\sin\frac{A}{2} \geq \frac{s}{R\sqrt{3}} \Leftrightarrow \sum_{cyc}\sin\frac{A}{2} \geq \frac{4}{\sqrt{3}}\prod_{cyc}\cos\frac{A}{2} \Leftrightarrow \sum_{cyc}\cos\left(\frac{\pi-A}{2}\right) \geq \frac{4}{\sqrt{3}}\prod_{cyc}\sin\left(\frac{\pi-A}{2}\right) \Leftrightarrow$$

(3) $\quad \sum_{cyc}\cos\alpha \geq \dfrac{4}{\sqrt{3}}\prod_{cyc}\sin\alpha$, where $\alpha := \dfrac{\pi-A}{2}, \beta := \dfrac{\pi-B}{2}, \gamma := \dfrac{\pi-C}{2}, \alpha,\beta,\gamma > 0$ and $\alpha+\beta+\gamma = \pi$.

Since $\sum_{cyc}\cos\alpha = 1+\dfrac{r}{R}$ and $\prod_{cyc}\sin\alpha = \dfrac{sr}{2R^2}$ then **(3)**$\Leftrightarrow 1+\dfrac{r}{R} \geq \dfrac{2sr}{\sqrt{3}R^2} \Leftrightarrow$

$$s \leq \frac{\sqrt{3}(R^2+Rr)}{2r} \Leftrightarrow F \leq \frac{\sqrt{3}(R^2+Rr)}{2}.$$

$$2rs \leq \sqrt{3}(R^2+Rr) \Leftarrow 3(R^2+Rr)^2 - 4(4R^2+4Rr+3r^2)r^2 =$$

$$(R-2r)(R+3r)(3Rr+3R^2+2r^2) \geq 0.$$